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Geometric Convergence of Lagrangian Interpolation and Numerical Integration Rules over Unbounded Contours and Intervals

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We investigate geometric convergence for rules of numerical integration and the associated Lagrangian interpolation polynomials over unbounded contours and intervals. The results obtained are shown to be substantially best possible.

1. MOTIVATION

Little is known about convergence rates of Lagrangian interpolation and numerical integration over unbounded intervals. Here elementary methods are used to investigate geometric convergence of not necessarily real rules of numerical integration and the associated Lagrangian interpolation polynomials. We show our results are substantially best possible. Further, in showing that our results contain some recent results of Aljarrah [1], we remove a restriction in his class of weights.

2. NOTATION

(i) Throughout, (a, b) will be a fixed unbounded real interval $(-\infty \leqslant a < b \leqslant \infty)$ and β: $(a, b) \to \mathbb{C}$ (the complex plane) will be continuous. Further, α: $(a, b) \to \mathbb{C}$ will be a fixed function of bounded variation such that the moment function

$$m(t) = \int_a^b |\beta(x)|^t \, d\,|\alpha|\,(x) \tag{2.1}$$

exists and is finite for all $t \geqslant 0$, as a Lebesgue–Stieltjes integral. Here

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$d|\alpha|(x)$ denotes the total variation of $d\alpha(x)$. To exclude "trivial" cases, we shall assume $\text{supp}[d|\alpha|]$ is infinite. Define also the moments

$$m_j = \int_a^b \beta(x)^j \, d\alpha(x), \qquad j = 0, 1, 2, \dots \tag{2.2}$$

Whenever the Lebesgue–Stieltjes integral is defined and finite, define

$$I[f(z)] = I[f] = \int_a^b f(\beta(x)) \, d\alpha(x) \tag{2.3}$$

The most important case of the above is where $\beta(x) = x$, and $\alpha(x)$ is real and monotone increasing, with $\text{supp}[d\alpha]$ unbounded. However, the above also includes complex integrals over unbounded contours.

(ii) Whenever $p > 0$ and the Lebesgue–Stieltjes integral is defined and finite, define

$$\| g \|_{\alpha,p} = \left\{ \int_a^b | g(\beta(x))|^p \, d|\alpha|(x) \right\}^{1/p}.$$

Also, whenever $r > 0$ and the sup is finite, set

$$\| g \|_{r,\infty} = \sup\{| g(z)|: |z| \leqslant r\}.$$

(iii) $I[f]$ will be approximated by

$$I_n[f] = \sum_{k=1}^{n} \lambda_{nk} f(x_{nk}), \qquad n = 1, 2, \dots \tag{2.4}$$

where the abscissas $x_{n1} \cdots x_{nn}$ are n distinct complex numbers and the weights $\lambda_{n1} \cdots \lambda_{nn}$ are complex numbers. Let

$$\Lambda(x) = \max \left\{ \sum_{k=1}^{n} |\lambda_{nk}|: 1 \leqslant n \leqslant x \right\} \qquad \text{all real } x \geqslant 1 \tag{2.5}$$

and

$$\Lambda = \limsup_{n \to \infty} \Lambda(n)^{1/n}. \tag{2.6}$$

Of course $\Lambda = 1$ for rules with positive real weights, but $\Lambda = 2$ for Newton–Cotes rules on $[-1, 1]$. See [11, p. 274]. Let

$$\tilde{\phi}_n(z) = \prod_{i=1}^{n} (z - x_{ni}), \qquad n = 1, 2, \dots \tag{2.7}$$

$$\gamma_n^{-1} = \|\tilde{\phi}_n\|_{\alpha,2}, \qquad n = 1, 2, \ldots \tag{2.8}$$

$$\phi_n(z) = \gamma_n \tilde{\phi}_n(z), \qquad n = 1, 2, \ldots. \tag{2.9}$$

In the special case where $\beta(x) = x$, and $\alpha(x)$ is real and monotone increasing, with $\text{supp}|d\alpha|$ unbounded, and where the $\{x_{ni}\}$ are the Gauss–Jacobi abscissas for $d\alpha$, then $\{\phi_n\}$ is the sequence of orthonormal polynomials for $d\alpha$, and $\{\gamma_n\}$ is the sequence of leading coefficients.

(iv) We assume that we are given a function $\Delta(n)$ bounding I_n's abscissa of largest modulus. More precisely, assume

$$\left.\begin{array}{l} \Delta: [1, \infty) \to (0, \infty) \text{ is monotone increasing} \\ \text{and } \lim_{x \to \infty} \Delta(x) = \infty \end{array}\right\} \tag{2.10}$$

and

$$\max\{|x_{nk}|: 1 \leqslant k \leqslant n\} \leqslant \Delta(n), \qquad n = 1, 2, \ldots. \tag{2.11}$$

For all $s > 0$, let

$$\psi(s) = \limsup_{x \to \infty} \Delta(xs)/\Delta(x) \tag{2.12}$$

$$\chi(s) = \limsup_{x \to \infty} \Delta(x + s)/\Delta(x). \tag{2.13}$$

We shall usually assume that $\chi(s)$ is finite for all $s \geqslant 1$. This allows $\Delta(x)$ to grow smoothly like x^η or $e^{\eta x}$ some $\eta > 0$.

(v) We shall assume I_n is exact for some powers of z, that is,

$$I_n[z^k] = I[z^k] = m_k, \qquad k = 0, 1, 2, \ldots, \kappa(n), \quad n = 1, 2, \ldots. \tag{2.14}$$

Regarding $\{\kappa(n)\}$, we assume throughout that $\{\kappa(n)\}$ is positive, monotone increasing and $\lim_{n \to \infty} \kappa(n) = \infty$. We shall often assume that for some fixed real $\zeta > , u \geqslant 0$,

$$\kappa(n) \geqslant \zeta n - u \qquad \text{all large enough } n \tag{2.15A}$$

and/or

$$\kappa(n) \leqslant \zeta n + u \qquad \text{all large enough } n. \tag{2.15B}$$

Let $\theta(v)$ be the "lower inverse" of $\kappa(n)$, that is,

$$\theta(x) = \min\{n: \kappa(n) \geqslant x\} \qquad \text{all real } x \geqslant \kappa(1) \tag{2.16}$$

so that $\theta(x)$ is monotone increasing, integer valued and

$$\kappa(\theta(x) - 1) < x \leqslant \kappa(\theta(x)) \qquad \text{all } x \geqslant \kappa(1). \tag{2.17}$$

Hence we have also

$$(2.15\text{A}) \Rightarrow \theta(x) < (x+u)/\zeta + 1 \qquad \text{all large enough } x \qquad (2.18\text{A})$$

$$(2.15\text{B}) \Rightarrow \theta(x) > (x-u)/\zeta \qquad \text{all large enough } x. \qquad (2.18\text{B})$$

For general interpolatory quadrature rules, $\kappa(n) = n - 1$; $\theta(x) =$ least integer $\geqslant x + 1$; $\zeta = u = 1$. Further, for Gauss–Jacobi rules $\kappa(n) = 2n - 1$; $\theta(x) =$ least integer $\geqslant (x+1)/2$; $\zeta = 2$; $u = 1$. Note, however, that we did not require ζ to be an integer in (2.15A, B).

(vi) Given real $p > 0$, we shall write throughout $p^* = \max\{1, p/\zeta\}$, without further mention. Here ζ is the number in (2.15A, B). Define

$$\left.\begin{aligned} \mu_+(p) &= \limsup_{n\to\infty} \|\tilde{\phi}_n\|_{\alpha,p}^{1/n}/\Delta(p^*n) \\ \mu_-(p) &= \liminf_{n\to\infty} \|\tilde{\phi}_n\|_{\alpha,p}^{1/n}/\Delta(p^*n) \end{aligned}\right\} \qquad (2.19)$$

$$\nu = \limsup_{n\to\infty} |(I - I_n)[z^{\kappa(n)+1}]|^{1/(\kappa(n)+1)}/\Delta(n). \qquad (2.20)$$

The quantities $\mu_+(p)$, $\mu_-(p)$ and ν depend only on the rules $\{I_n\}$ and are important in discussing geometric convergence.

(vii) The Lagrangian interpolation polynomial of order n ($n = 1, 2, \ldots$) to a function $f(z)$ defined at x_{ni}, $i = 1, 2, \ldots, n$ is

$$L_n[f](z) = \sum_{k=1}^{n} f(x_{nk})\, l_{nk}(z)$$

where

$$l_{nk}(z) = \prod_{\substack{j=1 \\ j\neq k}}^{n} \{(z - x_{nj})/(x_{nk} - x_{nj})\}, \qquad k = 1, 2, \ldots, n.$$

$\mathscr{P}_n$ will denote the class of polynomials of degree at most n with complex coefficients.

(viii) For any entire function $f(z) = \sum_{j=0}^{\infty} b_j z^j$, its order is

$$\rho(f) = \limsup_{n\to\infty} (n \log n)/(\log |b_n|^{-1}) \qquad (2.21)$$

and if $0 < \rho(f) < \infty$, its type is

$$\tau(f) = \limsup_{n\to\infty} n\, |b_n|^{\rho(f)/n}/(e\rho(f)). \qquad (2.22)$$

Finally, we introduce the Δ-index of f, denoted $\sigma(f; s)$, by

$$\sigma(f; s) = \limsup_{n \to \infty} |b_n|^{1/n} \Delta(sn) \qquad \text{all } s > 0. \tag{2.23}$$

For suitable values of s, $\sigma(f; s)$ determines whether or not $I_n[f]$ converges geometrically to $I[f]$; and whether or not $L_n[f]$ converges geometrically to f in some sense.

3. Lemmas

In this section, we establish some preliminary results. The following lemma compares ρ, τ and σ.

Lemma 3.1. *Let $f(z) = \sum_{j=0}^{\infty} b_j z^j$ be entire.*

(a) *Suppose for some $q_2 > 0$, $c_2 > 0$,*

$$\Delta(x) \leqslant (c_2 x)^{q_2} \qquad \textit{all large enough } x. \tag{3.1}$$

Then

(i) $\rho(f) < 1/q_2 \Rightarrow \sigma(f; s) = 0$ *all* $s > 0$.

(ii) $\rho(f) = 1/q_2 \Rightarrow \sigma(f; s) \leqslant (c_2 se\tau(f)/q_2)^{q_2}$ *all* $s > 0$.

(b) *Suppose for some $q_1 > 0$, $c_1 > 0$,*

$$\Delta(x) \geqslant (c_1 x)^{q_1} \qquad \textit{all large enough } x.$$

Then

(i) $\rho(f) > 1/q_1 \Rightarrow \sigma(f; s) = \infty$ *all* $s > 0$.

(ii) $\rho(f) = 1/q_1 \Rightarrow \sigma(f; s) \geqslant (c_1 se\tau(f)/q_1)^{q_1}$ *all* $s > 0$.

Proof. (a)(i) It follows from (2.21) that for some $q > q_2$ and for large n, $|b_n| \leqslant n^{-nq}$, so (3.1) gives

$$|b_n|^{1/n} \Delta(sn) \leqslant (c_2 s)^{q_2} n^{q_2 - q} \to 0 \qquad \text{as} \quad n \to \infty.$$

(ii) Writing $\rho = \rho(f) = 1/q_2$, we have

$$|b_n|^{1/n} \Delta(sn) \leqslant (c_2 s |b_n|^{\rho/n} n)^{q_2}$$

and (2.22), (2.23) give the result by taking lim sup's.

(b) is similar. Q.E.D.

One can introduce a "lower Δ-index" $\sigma_-(f; s)$ by replacing the lim sup in

(2.23) by lim inf. This is useful in some counterexamples, but we omit the details.

LEMMA 3.2. *Let* $\psi(s)$ *and* $\chi(s)$ *be given by* (2.12) *and* (2.13), *respectively.*

(a) $\psi(rs) \leqslant \psi(r)\,\psi(s)$; $\chi(r+s) \leqslant \chi(r)\,\chi(s)$ *all* $r, s > 0$.

(b) *If for some* $r > 1$, $\psi(r) < \infty$, *then* $\psi(s)$ *is finite for all* $s > 0$. *Further, given* $\varepsilon > 0$, *and if*

$$q_2 = (\log(\psi(r) + \varepsilon))/\log r \tag{3.2}$$

then for all large enough x,

$$\Delta(x) \leqslant (c_2 x)^{q_2}$$

where c_2 *is a positive constant.*

(c) *If for some* $r < 1$, $\psi(r) < 1$, *then* $\lim_{s \to 0+} \psi(s) = 0$. *Further, given small* $\varepsilon > 0$, *and if*

$$q_1 = (\log(\psi^{-1}(r) - \varepsilon))/\log r$$

then for all large enough x,

$$\Delta(x) \geqslant (c_1 x)^{q_1}$$

where c_1 *is a positive constant.*

Proof. (a) follows directly from (2.12) and (2.13). Note that we have to interpret $0 \cdot \infty = \infty$ if $\psi(r) = 0$, $\psi(s) = \infty$.

(b) From (a), $\psi(r^j) \leqslant (\psi(r))^j$ all $j = 1, 2, \ldots$ so $\psi(s)$ is finite for all $s > 0$. Next, for some positive integer l, we have

$$\Delta(rx)/\Delta(x) \leqslant \psi(r) + \varepsilon \qquad \text{all } x \geqslant r^l. \tag{3.3}$$

Then given $x \geqslant r^l$, we have for some integer $j \geqslant 0$,

$$r^{l+j-1} \leqslant x < r^{l+j} \Rightarrow l + j - 1 \leqslant \log x/\log r < l + j$$

so that by (3.3),

$$\begin{aligned} \Delta(x) \leqslant \Delta(r^{l+j}) &\leqslant \Delta(r)(\psi(r) + \varepsilon)^j \\ &\leqslant \Delta(r)(\psi(r) + \varepsilon)^{(\log x/\log r) + 1 - l} \\ &= (c_2 x)^{q_2} \end{aligned}$$

where q_2 is given by (3.2) and c_2 is a constant independent of x.

(c) is similar to (b). Q.E.D.

LEMMA 3.3. (a) $\|I_n[z^j]\| \leqslant \Lambda(n)\,\Delta^j(n), j, n = 0, 1, 2, \ldots$.

(b) $\|I[z^j]\| \leqslant \Lambda(\theta(j))\,\Delta^j(\theta(j)), j = 0, 1, 2, \ldots$.

(c) *When $\beta(x)$, $\alpha(x)$ are real valued functions and $d\alpha(x) \geqslant 0$ in (a, b), then*

$$m(t) \leqslant 2\Lambda(\theta(t+2))\,\Delta^t(\theta(t+2)) \qquad \text{all } t \geqslant 0.$$

Proof. (a) follows from (2.5) and (2.11).

(b) From (2.14), (2.17) we have $\kappa(\theta(j)) \geqslant j$ and hence $I[z^j] = I_{\theta(j)}[z^j]$. Now apply (a).

(c) Given $t \geqslant 0$, let j be the least even integer $\geqslant t$. Let K be an arbitrary positive number. Then if $\mathscr{S} = \{x\colon |\beta(x)| \leqslant K\}$,

$$\begin{aligned} m(t) &= \int_a^b |\beta(x)|^t \, d\alpha(x) \\ &\leqslant K^t \int_{\mathscr{S}} d\alpha(x) + \int_{(a,b)\setminus\mathscr{S}} |\beta(x)|^t (|\beta(x)|/K)^{j-t} \, d\alpha(x) \\ &\leqslant K^t m_0 + K^{t-j} m_j, \end{aligned}$$

as $\beta(x)$ is real and j is even. Choose $K = \Delta(\theta(t+2))$ and use the bound in (b) together with monotonicity of $\Lambda(x)$, $\Delta(x)$, $\theta(x)$ and $j < t+2$. Q.E.D.

LEMMA 3.4. *Let $j \geqslant n \geqslant 1$ and $P(z) = z^j - L_n[z^j](z)$. Let $0 < \delta < 1$ and $h(\delta) = 2\delta/(1-\delta)$.*

(a) $\Delta^j(n) \leqslant \|P\|_{\Delta(n),\infty} \leqslant h^{n+1}(\delta)(\Delta(n)/\delta)^j/(2\delta)$.

(b) *When $j = n$, $P(z) = \tilde{\phi}_n(z)$.*

(c) *When $\beta(x)$, $\alpha(x)$ are real valued functions and $d\alpha(x) \geqslant 0$ in (a, b), then for $p \geqslant 1$,*

$$\|P\|_{\alpha,p} \leqslant h^{n+1}(\delta)\,\Lambda^{1/p}(\theta(jp+2))(\Delta(l)/\delta)^j(2/\delta)$$

where $l = \max\{n, \theta(jp+2)\}$.

Proof. (a) The left part of the inequality follows from the well-known fact that for any monic polynomial $Q(z)$ of degree j, $\max\{|Q(z)|\colon |z| \leqslant r\} \geqslant r^j$—see, for example, Hille [9, p. 267]. The upper bound follows by using

$$P(z) = z^j - L_n[z^j](z) = (2\pi i)^{-1} \int_C \frac{t^j}{(t-z)} \frac{\tilde{\phi}_n(z)}{\tilde{\phi}_n(t)} \, dt$$

all $|z| \leqslant \Delta(n)$, where $C = \{t\colon |t| = \Delta(n)/\delta\}$—see Davis [3, Theorem 3.6.1,

p. 68]. Since the zeroes of $\tilde{\phi}_n$ lie in $|z| \leqslant \Delta(n)$, estimating the contour integral in the usual way yields the right part of the inequality.

(b) Both $P(z) = z^n - L_n[z^n](z)$ and $\tilde{\phi}_n(z)$ have leading coefficient 1, degree n, and the n zeroes x_{nk}, $k = 1, 2, \ldots, n$, in common, so must be identical.

(c) Let $r > 0$ and $\mathscr{S} = \{x \in (a, b): |\beta(x)| \leqslant r\}$. Then for $x \in \mathscr{S}$, $|P(\beta(x))| \leqslant \|P\|_{r,\infty}$, and for $x \in (a, b)\backslash\mathscr{S}$, $|P(\beta(x))| \leqslant \|P\|_{r,\infty} |\beta(x)/r|^j$ by the Walsh–Bernstein inequality [10, p. 77]. Thus

$$\begin{aligned}\|P\|_{\alpha,p}^p &\leqslant \int_{\mathscr{S}} \|P\|_{r,\infty}^p \, d\alpha(x) + \int_{(a,b)\backslash\mathscr{S}} \|P\|_{r,\infty}^p \, |\beta(x)/r|^{jp} \, d\alpha(x) \\ &\leqslant \|P\|_{r,\infty}^p \, (m_0 + m(jp)/r^{jp}).\end{aligned}$$

Taking $r = \Delta(n)$ and using Lemma 3.3(c), we see

$$\begin{aligned}m_0 + m(jp)/r^{jp} &\leqslant A(\theta(0)) + 2A(\theta(jp+2)) \, \Delta^{jp}(\theta(jp+2))/\Delta^{jp}(n) \\ &\leqslant 3A(\theta(jp+2))(\Delta(l)/\Delta(n))^{jp}\end{aligned}$$

by definition of l. If we now use the upper bound on $\|P\|_{\Delta(n),\infty}$ in (a) above, and take pth roots, the result follows. Q.E.D.

Lemmas 3.4(a), (b) remain true when some of the abscissas x_{ni} coalesce. Lemma 3.4(c) remains true with a suitably modified definition of $A(n)$.

LEMMA 3.5. *Let* $f(z) = \sum_{j=0}^{\infty} b_j z^j$ *be entire. Assume* A *given by* (2.6) *is finite and* $\chi(s)$ *given by* (2.13) *is finite for* $s > 0$. *Set* $h(x) = 2x/(1-x)$ *all* $0 < x < 1$. *Assume that* (2.15A) *holds.*

(a) *Let* $c_0 = \chi(u/\zeta + 1) \, A^{1/\zeta}$. *If* $\sigma(f; 1/\zeta) < 1/c_0$ *then*

$$\limsup_{n\to\infty} |(I - I_n)[f]|^{1/n} \leqslant (c_0 \sigma(f; 1/\zeta))^{\zeta} < 1.$$

(b) *If* $\sigma(f; 1) < 1/3$, *then*

$$\limsup_{n\to\infty} \|f - L_n[f]\|_{\Delta(n),\infty}^{1/n} \leqslant h(\sigma(f; 1)) < 1.$$

(c) *Assume* $\beta(x)$, $\alpha(x)$ *are real valued functions and* $d\alpha(x) \geqslant 0$ *in* (a, b). *Let* $p \geqslant 1$ *and* $c_1 = \chi((u+2)/\zeta + 1) \, A^{1/\zeta}$. *If* $\sigma(f; p^*) < 1/(3c_1)$ *then*

$$\limsup_{n\to\infty} \|f - L_n[f]\|_{\alpha,p}^{1/n} \leqslant h(c_1 \sigma(f; p^*)) < 1.$$

Proof. (a) Using (2.14), we have (at first formally)

$$(I - I_n)[f] = \sum_{j=\kappa(n)+1}^{\infty} b_j(I - I_n)[z^j]. \tag{3.4}$$

Now $j > \kappa(n) \Rightarrow \theta(j) > n$ by (2.17). Further, (2.18A) holds. These remarks together with Lemmas 3.3(a), (b) and monotonicity of $A(x)$, $\Delta(x)$ give, for all $j > \kappa(n)$,

$$|(I - I_n)[z^j]| \leqslant 2A((j+u)/\zeta + 1)\, \Delta^j((j+u)/\zeta + 1).$$

Then

$$\begin{aligned}
&\limsup_{n\to\infty} \{ \max_{j>\kappa(n)} |b_j(I - I_n)[z^j]|^{1/j} \} \\
&\quad \leqslant \limsup_{n\to\infty} \{ \max_{j>\kappa(n)} A^{1/j}((j+u)/\zeta + 1)[\Delta((j+u)/\zeta + 1)/\Delta(j/\zeta)] \\
&\qquad \times [|b_j|^{1/j} \Delta(j/\zeta)]\} \\
&\quad \leqslant A^{1/\zeta}\chi(u/\zeta + 1)\, \sigma(f;\, 1/\zeta) = c_0 \sigma(f;\, 1/\zeta)
\end{aligned}$$

by (2.6), (2.13) and (2.23). Then given small $\varepsilon > 0$, (2.15A) and (3.4) give, for large enough n,

$$\begin{aligned}
|(I - I_n)[f]| &\leqslant \sum_{j=\zeta n - u + 1}^{\infty} ((1+\varepsilon)\, c_0 \sigma(f;\, 1/\zeta))^j \\
&\leqslant K((1+\varepsilon)\, c_0 \sigma(f;\, 1/\zeta))^{\zeta n}
\end{aligned}$$

where K is a constant independent of n. The result follows.

(b) We have

$$(f - L_n[f])(z) = \sum_{j=n}^{\infty} b_j(z^j - L_n[z^j](z)). \tag{3.5}$$

Choose $\varepsilon > 0$ such that $d = \sigma(f;\, 1) + \varepsilon$ and $\delta = \sigma(f;\, 1) + 2\varepsilon$ are less than 1. Then by Lemma 3.4(a), for large n,

$$\begin{aligned}
\|f - L_n[f]\|_{\Delta(n),\infty} &\leqslant h^{n+1}(\delta)(2\delta)^{-1} \sum_{j=n}^{\infty} |b_j|\, (\Delta(n)/\delta)^j \\
&\leqslant h^{n+1}(\delta)(2\delta)^{-1} \sum_{j=n}^{\infty} (d/\delta)^j
\end{aligned}$$

by (2.23) and monotonicity of $\Delta(x)$. The result follows as δ and $d < \delta$ can be chosen arbitrarily close to $\sigma(f;\, 1)$.

(c) Choose ε so small that $d = c_1 \sigma(f; p^*) + \varepsilon$ and $\delta = c_1 \sigma(f; p^*) + 2\varepsilon$ are smaller than 1. Then using the upper bound in (2.18A), as well as Lemma 3.4(c), and (3.5), we obtain (at first formally)

$$\|f - L_n[f]\|_{\alpha,p} \leqslant 2h^{n+1}(\delta)\,\delta^{-1} \sum_{j=n}^{\infty} \Delta^{1/p}((jp + 2 + u)/\zeta + 1)\,|b_j|$$

$$\times (\Delta(jp^* + (2+u)/\zeta + 1)/\delta)^j$$

$$\leqslant h^{n+1}(\delta)\,K \sum_{j=n}^{\infty} (d/\delta)^j$$

for large n by choice of c_1 and d, and where K is a constant independent of n. The result follows. Q.E.D.

LEMMA 3.6. *Let $f(z) = \sum_{j=0}^{\infty} b_j z^j$ be entire with b_j real, $j = 0, 1, 2,\ldots$. Let n be a given positive integer and x_{nj}, $j = 1, 2,\ldots, n$, be real.*

(a) $|f - L_n[f]|\,(x) \leqslant |\tilde{\phi}_n(x)| \sum_{j=n}^{\infty} \binom{j}{n} |b_j|\,(\max\{|x|, \Delta(n)\})^{j-n}$, *for all real x.*

(b) *If* $b_{2j} \geqslant 0$, $b_{2j+1} = 0$, $j = 0, 1, 2,\ldots,$ *then* $|f - L_{2n}[f]|\,(x) \geqslant b_{2n}\,|\tilde{\phi}_{2n}(x)|$ *for all real x.*

(c) *If* $b_{2j} = 0$, $b_{2j+1} \geqslant 0$, $j = 0, 1, 2,\ldots,$ *then* $|f - L_{2n+1}[f]|\,(x) \geqslant b_{2n+1}\,|\tilde{\phi}_{2n+1}(x)|$ *for all real x.*

(d) *If* $x_{nj} \geqslant 0$, $j = 1, 2,\ldots, n$ *and* $b_j \geqslant 0$, $j = 0, 1, 2,\ldots,$ *then* $|f - L_n[f]|(x) \geqslant b_n\,|\tilde{\phi}_n(x)|$ *for all $x \geqslant 0$.*

Proof. By the well-known error formula for Lagrange interpolation (Davis [3, Theorem 3.1.1, p. 56]) for all real x,

$$(f - L_n[f])(x) = f^{(n)}(\eta)\,\tilde{\phi}(x)/n!$$

$$= \tilde{\phi}_n(x) \sum_{j=n}^{\infty} \binom{j}{n} b_j \eta^{j-n}. \tag{3.6}$$

Here η depends on x and is contained in the smallest interval containing x_{nk}, $k = 1, 2,\ldots, n$, and x.

(a) follows from (3.6) and $|\eta| \leqslant \max\{\Delta(n), |x|\}$.

(b) Using (3.6) and the restrictions on $\{b_j\}$,

$$|f - L_{2n}[f]|\,(x) = |\tilde{\phi}_{2n}(x)| \sum_{j=n}^{\infty} \binom{2j}{2n} b_{2j}\,\eta^{2j-2n}$$

$$\geqslant |\tilde{\phi}_{2n}(x)|\,b_{2n}.$$

(c), (d) are similar to (b). Q.E.D.

LEMMA 3.7. *Assume that $\beta(x)=x$ and that $\alpha(x)$ is real and absolutely continuous in (a,b). Assume that $d\alpha(x)\geqslant 0$ in (a,b). Further, assume there exists $\delta\in(0,1)$ satisfying*

$$\liminf_{n\to\infty} \operatorname{meas}\{x\in(a,b)\colon \alpha'(x)\geqslant\delta^n\}/\Delta(n)>0 \tag{3.7}$$

where meas *denotes linear Lebesgue measure. Then $\mu_-(p)$ given by* (2.19) *is positive for all $p>0$.*

Proof. Let $p>0$. Let $\mathscr{S}(n)=\{x\in(a,b)\colon \alpha'(x)\geqslant\delta^{2p^*n}\}$ all $n=1,2,\ldots$. For large n, there is an integer between p^*n and $2p^*n$, so monotonicity of $\Delta(x)$ and (3.7) yield

$$\operatorname{meas}(\mathscr{S}(n))\geqslant c\Delta(p^*n) \qquad \text{all large enough } n \tag{3.8}$$

where $c>0$ is independent of n Choose $0<\varepsilon<c/(4e)$ and let $\mathscr{L}(n)=\{x\colon|\tilde{\phi}_n(x)|\leqslant(\varepsilon\Delta(p^*n))^n\}$, $n=1,2,\ldots$. By Cartan's Lemma (Baker [2, p. 174])

$$\operatorname{meas}(\mathscr{L}(n))\leqslant 4e\varepsilon\Delta(p^*n), \qquad n=1,2,\ldots. \tag{3.9}$$

Then by absolute continuity of $\alpha(x)$,

$$\begin{aligned}\|\tilde{\phi}\|^p_{\alpha,p}&=\int_a^b|\tilde{\phi}_n(x)|^p\,\alpha'(x)\,dx\\&\geqslant\int_{\mathscr{S}(n)\setminus\mathscr{L}(n)}(\varepsilon\Delta(p^*n))^{np}\,\delta^{2p^*n}\,dx\\&\geqslant\Delta^{np}(p^*n)(\varepsilon^p\delta^{2p^*})^n((c-4e\varepsilon)\,\Delta(p^*n))\end{aligned}$$

by (3.8), (3.9). The result follows as $c-4e\varepsilon>0$. Q.E.D.

LEMMA 3.8. *Assume that $\beta(x)=x$ and that $\alpha(x)$ is real and $d\alpha(x)\geqslant 0$ in (a,b), with* supp$|d\alpha|$ *unbounded. Assume that the $\{\lambda_{ni}\}$ and $\{x_{ni}\}$ are, respectively, the Gauss–Jacobi weights and abscissas for $d\alpha(x)$.*

(a) *Let $f(z)=\sum_{j=0}^{\infty}b_{2j}z^{2j}$ be entire with $b_{2j}\geqslant 0$, $j=0,1,2,\ldots$. Then*

$$I[f]<\infty\Leftrightarrow\sum_{j=0}^{\infty}b_{2j}m_{2j}<\infty \tag{3.10}$$

and if either of these holds,

$$(I-I_n)[f]\geqslant b_{2n}\gamma_n^{-2} \qquad \textit{all } n\geqslant 1. \tag{3.11}$$

(b) *If v is given by* (2.20), *then $v=\mu_+(2)$.*

(c) *If* $g(z)$ *is defined at* x_{nj}, $j = 1, 2, \ldots, n$, *then*

$$\|L_n[g]\|_{\Delta(n),\infty} \leqslant \gamma_{n-1}(2\Delta(n))^{n-1} m_0^{1/2} \max_{1 \leqslant k \leqslant n} |g(x_{nk})|. \tag{3.12}$$

Proof. (a) Since the partial sums of f increase monotonically to f in $(-\infty, \infty)$, (3.10) follows from Lebesgue's monotone convergence theorem. To show (3.11), note first that $(I - I_n)[x^{2j}] \geqslant 0$ all $j = 0, 1, 2, \ldots$ by Shohat's Lemma (Freud [8, Lemma III.1.5, p. 92]) and hence

$$\begin{aligned}(I - I_n)[f] &= \sum_{j=n}^{\infty} b_{2j}(I - I_n)[x^{2j}] \\ &\geqslant b_{2n}(I - I_n)[x^{2n}].\end{aligned}$$

Next, let $H(x)$ be the (Hermite) interpolation polynomial of degree at most $2n - 1$ that interpolates to the value and first derivative of x^{2n} at $x = x_{nj}$, $j = 1, 2, \ldots, n$. Then the polynomials $z^{2n} - H(z)$ and $(\tilde{\phi}_n(z))^2$ have leading coefficient one, and $2n$ zeroes in common, so must be identical. Thus by exactness of the quadrature rule and by (2.8),

$$\begin{aligned}(I - I_n)[x^{2n}] &= I[x^{2n}] - I_n[H] \\ &= I[x^{2n}] - I[H] \\ &= I[\tilde{\phi}_n^2] = \gamma_n^{-2}.\end{aligned} \tag{3.13}$$

(b) As $\kappa(n) = 2n - 1$ for Gauss–Jacobi rules, (2.20) yields

$$\begin{aligned}v &= \limsup_{n \to \infty} |(I - I_n)[x^{2n}]|^{1/(2n)}/\Delta(n) \\ &= \limsup_{n \to \infty} \|\tilde{\phi}_n\|_{\alpha,2}^{1/n}/\Delta(n) = \mu_+(2)\end{aligned}$$

by (3.13), (2.8) and as $2^* = \max\{1, 2/\zeta\} = 1$ for Gauss–Jacobi rules, since $\zeta = 2$ in (2.15A).

(c) By Eq. III.(6.3) in Freud [8, p. 114],

$$l_{nk}(z) = (\gamma_{n-1}/\gamma_n)\, \lambda_{nk} \phi_{n-1}(x_{nk})\, \phi_n(z)/(z - x_{nk}), \qquad k = 1, 2, \ldots, n.$$

(His notation is a little different from that here.) Then if I_n acts on the variable y, we see from (2.9) that

$$L_n[g](z) = \gamma_{n-1} \tilde{\phi}_n(z)\, I_n[\phi_{n-1}(y)\, g(y)/(z - y)].$$

Using Holder's inequality and $I_n|\phi_{n-1}^2(y)| = I|\phi_{n-1}^2| = 1$, we obtain

$$\|L_n|g|\|_{\Delta(n),\infty} \leqslant \gamma_{n-1} \max_{1\leqslant k\leqslant n} \|\tilde{\phi}_n(z)/(z-x_{nk})\|_{\Delta(n),\infty} I_n^{1/2}\|g|^2|$$

and (3.12) follows. Q.E.D.

When $(a, b) \subset (0, \infty)$ an obvious analogue of Lemma 3.8(a) holds for entire $f(z) = \sum_{j=0}^{\infty} b_j z^j$ with $b \geqslant 0, j = 0, 1, 2, \ldots$.

4. General Theorems

Following is our main result.

THEOREM 4.1. *Assume A given by* (2.6) *is finite and that $\chi(s)$ given by* (2.13) *is finite for all $s > 0$. Write $h(x) = 2x/(1-x)$ for all $0 < x < 1$. Assume* (2.15A) *holds.*

(a) *Let $c_0 = \chi(u/\zeta + 1) A^{1/\zeta}$. For any entire function f such that* $\sigma(f; 1/\zeta) < 1/c_0$, we have

$$\limsup_{n\to\infty} |(I - I_n)|f|\|^{1/n} \leqslant (c_0 \sigma(f; 1/\zeta))^{\zeta}. \tag{4.1}$$

(b) *For any entire function f such that $\sigma(f; 1) < 1/3$, we have*

$$\limsup_{n\to\infty} \|f - L_n|f|\|_{\Delta(n),\infty}^{1/n} \leqslant h(\sigma(f; 1)). \tag{4.2}$$

(c) *Assume $\beta(x)$, $\alpha(x)$ are real valued functions and $d\alpha(x) \geqslant 0$ in (a, b). Let $p \geqslant 1$ and $c_1 = \chi((u+2)/\zeta + 1) A^{1/\zeta}$. For any entire function f such that $\sigma(f; p^*) < 1/(3c_1)$, we have*

$$\limsup_{n\to\infty} \|f - L_n|f|\|_{\alpha,p}^{1/n} \leqslant h(c_1 \sigma(f; p^*)). \tag{4.3}$$

In addition, let (2.15B) *hold. We then have the following negative assertions to complement the above:*

(a′) *There exists an entire function $f(z)$ with $\sigma(f; 1/\zeta)$ an arbitrary number in $(0, 1/c_0)$ and*

$$\limsup_{n\to\infty} |(I - I_n)|f|\|^{1/n} \geqslant (v\sigma(f; 1/\zeta)/\chi((u+1)/\zeta))^{\zeta} \tag{4.4}$$

where v is given by (2.20).

(b′) *There exists an entire function* $f(z)$ *with* $\sigma(f; 1)$ *an arbitrary number in* $(0, 1/3)$ *and*

$$\limsup_{n\to\infty} \|f - L_n[f]\|_{\Delta(n),\infty}^{1/n} \geqslant \sigma(f; 1). \tag{4.5}$$

(c′) *Let* $p \geqslant 1$. *There exists an entire function* $f(z)$ *with* $\sigma(f; p^*)$ *an arbitrary number in* $(0, 1/(3c_1))$ *and*

$$\limsup_{n\to\infty} \|f - L_n[f]\|_{\alpha,p}^{1/n} \geqslant \mu_+(p)\,\sigma(f; p^*). \tag{4.6}$$

Proof. The positive assertions (4.1), (4.2), (4.3) follow immediately from Lemma 3.5. We need prove only the counterexamples by choosing suitable entire functions $f(z) = \sum_{j=0}^{\infty} b_j z^j$.

(a′) Assume $\nu > 0$, otherwise the counterexample is trivial. Let $\sigma \in (0, 1/c_0)$. By (2.20), we can choose positive integers $n(i)$ with the following property. Let $k(i) = \kappa(n(i)) + 1$, $i = 1, 2,\ldots$. Then, for $i = 1, 2,\ldots$,

$$k(i+1) > (i+1)\,k(i) \tag{4.7}$$

$$n = n(i), \qquad k = k(i) \Rightarrow \|(I - I_n)[z^k]\| \geqslant |(\nu - 1/i)\,\Delta(n)|^k. \tag{4.8}$$

Let $\mathscr{K} = \{k(i) : i = 1, 2,\ldots\}$. Set $b_k = 0$ if $k \notin \mathscr{K}$ and

$$b_k \Delta^k(k/\zeta) = \sigma^k \qquad \text{if} \quad k \in \mathscr{K}. \tag{4.9}$$

Then $f(z) = \sum_{j=0}^{\infty} b_j z^j$ is entire and clearly $\sigma(f; 1/\zeta) = \sigma$. Further, if $n = n(i)$, $k = k(i)$, (3.4), (4.8) and (4.9) give

$$\begin{aligned} \|(I - I_n)[f]\| &= \Big\| \sum_{j\in\mathscr{K}, j\geqslant k} b_j (I - I_n)[z^j] \Big\| \\ &\geqslant |\sigma(\nu - 1/i)\,\Delta(n)/\Delta(k/\zeta)|^k - \sum_{j\in\mathscr{K}, j>k} |b_j|\,\|(I - I_n)[z^j]\|. \end{aligned} \tag{4.10}$$

Now if $k = k(i)$, $n = n(i)$, then $k \leqslant \zeta n + u + 1$ by (2.15B), so

$$\limsup_{i\to\infty} \Delta(k/\zeta)/\Delta(n) \leqslant \limsup_{n\to\infty} \Delta(n + (u+1)/\zeta)/\Delta(n) = \chi((u+1)/\zeta). \tag{4.11}$$

Choose small $\varepsilon > 0$. Proceeding as in the proof of Lemma 3.5(a), and using (4.7), (2.15A), we see that for large i, $n = n(i)$, $k = k(i)$,

$$\sum_{j\in\mathscr{K}, j>k} |b_j|\,\|(I - I_n)[z^j]\| \leqslant K((1+\varepsilon)\,c_0\,\sigma(f; 1/\zeta))^{i\zeta n} \tag{4.12}$$

where K is a constant independent of n, i. Then (4.4) follows from (4.10), (4.11), (4.12) by letting $i \to \infty$ and $n = n(i)$, $k = k(i)$.

(b′) Here one uses (3.5) and $\|z^n - L_n[z^n]\|_{\Delta(n),\infty} \geqslant \Delta^n(n)$ (by Lemma 3.4(a)) and chooses the $\{b_j\}$ much as in (a).

(c′) From (3.5), it follows that

$$\|f - L_n[f]\|_{\alpha,p} \geqslant |b_n| \, \|z^n - L_n[z^n]\|_{\alpha,p} - \sum_{j=n+1}^{\infty} |b_j| \, \|z^j - L_n[z^j]\|_{\alpha,p}$$

provided the norms and series are finite. Further, by Lemma 3.4(b),

$$|b_n| \, \|z^n - L_n[z^n]\|_{\alpha,p} = \{|\,|b_n|^{1/n} \Delta(p^*n)| \, |\|\tilde{\phi}_n\|_{\alpha,p}^{1/n}/\Delta(p^*n)|\}^n.$$

Using these equations, and (2.19), (2.23), one can choose the $\{b_j\}$ much as in (a′) to deduce (4.6). Q.E.D.

Remarks. (a) Theorems 4.1(b), (c) remain valid when some abscissas $\{x_{ni}\}$ coalesce, so that $L_n[f]$ interpolates to some derivatives of f. When they are interpolatory, the integration rules may also be modified to use f's derivatives: in studying convergence one then uses the inequality $|(I - I_n)[f]| \leqslant \|f - L_n[f]\|_{\alpha,1}$.

(b) Obviously the counterexamples in Theorems 4.1(a′), (c′) have no significance unless $\nu > 0$, $\mu_+(p) > 0$, respectively. It is possible to construct rules for which both these quantities are zero—in such cases it is perhaps inappropriate to base a study of geometric convergence on the function $\Delta(n)$. However, in practical rules, both ν and $\mu\,(p)$, and so $\mu_+(p)$, are positive—see Section 5.

(c) The negative assertions show that in each case, the term involving $\sigma(f; \cdot)$ substantially determines the rate of convergence. For example, (a), (a′) yield functions satisfying

$$K_1 \leqslant \limsup_{n\to\infty} |(I - I_n)[f]|^{1/n}/(\sigma(f; 1/\zeta))^{\zeta} \leqslant K_2$$

where K_1 and K_2 depend only on the integration rules, not on f, and where K_1 is usually positive.

COROLLARY 4.2. *Assume that Λ, $\chi(s)$ are as in Theorem* 4.1.

(a) *Assume that for some $q > 0$, $c_2 > 0$,*

$$\Delta(n) \leqslant (c_2 n)^q \qquad \textit{for all large enough } n. \tag{4.13}$$

If f is entire and $\rho(f) < 1/q$, then we may replace the right members of (4.1), (4.2) *and* (4.3) *by* 0. *If $\rho(f) = 1/q$ then we may replace $\sigma(f; s)$ in* (4.1), (4.2) *and* (4.3) *by $(c_2 se\tau(f)/q)^q$ with the appropriate values of s, provided s is small enough to satisfy the requirements of those assertions.*

(b) *Assume, in addition to* (4.13), *that for some* $c_3 > 0$,

$$\Delta(n) \geqslant (c_3 n)^q \qquad \textit{for all large enough } n.$$

Then there exist entire functions f *with* $\rho(f) = 1/q$ *and satisfying the restrictions on* $\sigma(f; s)$ *in* (4.4), (4.5) *and* (4.6), *with the appropriate values of* s. *Further, for those entire functions, we may replace* $\sigma(f; s)$ *in* (4.4), (4.5) *and* (4.6) *by* $(c_3 se\tau(f)/q)^q$ *with the appropriate values of* s.

Proof. This follows immediately from Lemma 3.1 and Theorem 4.1.

Q.E.D.

Remarks. (a) When $\psi(r) < \infty$, some $r > 1$, and $\varepsilon > 0$ is given, Lemma 3.2(b) shows we may choose $q = \inf\{\log(\psi(r) + \varepsilon)/\log r\colon\ r > 1\}$ in (4.13).

(b) When the abscissas $\{x_{ni}\}$ are real, there is the following alternative result for Lagrangian interpolation.

THEOREM 4.3. *Assume that* $\beta(x)$, $\alpha(x)$ *are real valued and* $d\alpha(x) \geqslant 0$ *in* (a, b). *Assume that* x_{ni}, $i = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ *are all real. Assume that* Λ *given by* (2.6) *is finite, and that* (2.15A) *holds.*

(a) *Let* $q_1 \geqslant p_1 > p \geqslant 1$ *and* $p_1^{-1} + q_1^{-1} = p^{-1}$. *Then if* f *is an entire function such that* $\sigma(f; q_1^*) < \Lambda^{-1/\zeta}$, *we have*

$$\limsup_{n \to \infty} \|f - L_n[f]\|_{\alpha, p}^{1/n} \leqslant \mu_+(p_1)\, \sigma(f; q_1^*)/(1 - \Lambda^{1/\zeta}\sigma(f; q_1^*)).$$

(b) *By contrast, if* $f(z) = \sum_{j=0}^{\infty} b_{2j} z^{2j}$ *is entire with* $b_{2j} \geqslant 0$, $j = 0, 1, 2, \ldots$, *then for* $p \geqslant 1$ *such that* $\|f\|_{\alpha, p} < \infty$, *we have*

$$\limsup_{n \to \infty} \|f - L_n[f]\|_{\alpha, p}^{1/n} \geqslant \mu_-(p)\, \sigma(f; p^*).$$

Proof. (a) Assume first $f(z) = \sum_{j=0}^{\infty} b_j z^j$ *is real for real* z. *Lemma* 3.6(a) *and Hölder's and Minkowski's inequalities give* (*at first formally*)

$$\|f - L_n[f]\|_{\alpha, p} \leqslant \|\tilde{\phi}_n\|_{\alpha, p_1} \sum_{j=n}^{\infty} \binom{j}{n} |b_j|\, \|(\max\{|x|, \Delta(n)\})^{j-n}\|_{\alpha, q_1}. \tag{4.14}$$

Here by Minkowski's inequality and (2.1),

$$\begin{aligned}
&\|(\max\{|x|, \Delta(n)\})^{j-n}\|_{\alpha, q_1} \\
&\quad \leqslant \||x|^{j-n}\|_{\alpha, q_1} + \|\Delta^{j-n}(n)\|_{\alpha, q_1} \\
&\quad = m^{1/q_1}((j-n)\, q_1) + m^{1/q_1}(0)\, \Delta^{j-n}(n) \\
&\quad \leqslant \{2\Lambda(\theta[(j-n)\, q_1 + 2])\}^{1/q_1} \{\Delta^{j-n}(\theta[(j-n)\, q_1 + 2]) + \Delta^{j-n}(n)\}
\end{aligned}$$

(by Lemma 3.3(c) and monotonicity of Λ)

$$\leqslant K\Lambda^{(j-n)/\zeta} \Delta^{j-n}(jq_1^*) \qquad \text{all } j \geqslant n, \text{ all } n \text{ large enough.} \tag{4.15}$$

Here K is a constant independent of n, and j, and we have used (2.6), (2.18A), the monotonicity of Δ, and the definition of q_1^*. Now let $\varepsilon > 0$ be so small that

$$d = \Lambda^{1/\zeta}\sigma(f; q_1^*) + \varepsilon < 1. \tag{4.16}$$

Using $q_1^* \geqslant p_1^*$ and (4.14), (4.15), we obtain, for large n,

$$\begin{aligned}\|f - L_n[f]\|_{\alpha,p} &\leqslant K\Lambda^{-n/\zeta} \|\tilde{\phi}_n\|_{\alpha,p}/\Delta^n(np_1^*) \sum_{j=n}^{\infty} \binom{j}{n} |b_j| \Lambda^{j/\zeta} \Delta^j(jq_1^*) \\ &\leqslant K(\Lambda^{-1/\zeta}(\mu_+(p) + \varepsilon))^n \sum_{j=n}^{\infty} \binom{j}{n} d^j \\ &= K(\Lambda^{-1/\zeta}(\mu_+(p) + \varepsilon))^n d^n(1-d)^{-n-1}.\end{aligned} \tag{4.17}$$

The result follows from (4.16) and (4.17) by taking nth roots and letting $n \to \infty$. When f is non-real for real z, write $f(z) = f_1(z) + if_2(z)$, where f_1, f_2 are entire and real for real z. Further, use the linearity of L_n as well as $\sigma(f_j; s) \leqslant \sigma(f; s)$, $j = 1, 2$; $s \geqslant 0$.

(b) By Lemma 3.6(b),

$$\|f - L_{2n}[f]\|_{\alpha,p}^{1/(2n)} \geqslant \{b_{2n}^{1/(2n)} \Delta(2p^*n)\}\{\|\tilde{\phi}_{2n}\|_{\alpha,p}^{1/(2n)}/\Delta(2p^*n)\}$$

and the result follows from (2.19), (2.23). Q.E.D.

Theorem 4.1(c) and Theorem 4.3(a) complement one another—neither contains the other in general.

5. Gauss–Jacobi Rules

Throughout this section—without further mention—we assume $\beta(x) = x$ and that $\alpha(x)$ is real and monotone increasing in (a, b) and that supp$|d\alpha|$ is unbounded. Further, we assume that the $\{\lambda_{ni}\}$ and $\{x_{ni}\}$ are, respectively, the Gauss–Jacobi weights and abscissas for $d\alpha(x)$. Thus $\{\phi_n\}$ given by (2.9) is the sequence of orthonormal polynomials for $d\alpha(x)$ and $\{\gamma_n\}$ is the sequence of leading coefficients.

THEOREM 5.1. *Assume $\chi(s)$ is finite for all $s > 0$.*

(a) *For any entire function f such that $\sigma(f; 1/2) < 1/\chi(3/2)$, we have*

$$\limsup_{n\to\infty} |(I-I_n)[f]|^{1/n} \leqslant (\chi(3/2)\,\sigma(f; 1/2))^2.$$

By contrast, given entire $f(z)=\sum_{j=0}^{\infty} b_{2j}z^{2j}$ with $b_{2j}\geqslant 0, j=0, 1, 2,...,$ and $I[f]<\infty$, we have

$$\limsup_{n\to\infty} |(I-I_n)[f]|^{1/n} \geqslant (\mu_-(2)\,\sigma(f; 1/2))^2.$$

(b) *For any entire function f, let*

$$E_{n-1}[f]=\min\{\|f-P\|_{\Delta(n),\infty} : P\in \mathscr{P}_{n-1}\}, \qquad n=1, 2,....$$

Then

$$1 \leqslant \liminf_{m} \{\|f-L_n[f]\|_{\Delta(n),\infty}/E_{n-1}[f]\}^{1/n}$$
$$\leqslant \limsup_{n\to\infty} \{\|f-L_n[f]\|_{\Delta(n),\infty}/E_{n-1}[f]\}^{1/n} \leqslant 2\chi(1)/\mu_-(2).$$

Proof. (a) The positive assertion follows from Theorem 4.1(a), as $\zeta=2$, $u=1$, $A=1$. By contrast, if $f(z)=\sum_{j=0}^{\infty} b_{2j}z^{2j}$ is entire with $b_{2j}\geqslant 0$, $j=0, 1, 2,...,$ and $I[f]<\infty$, Lemma 3.8(a) gives

$$\{(I-I_n)[f]\}^{1/n} \geqslant b_{2n}^{1/n}\gamma_n^{-2/n} = \{b_{2n}^{1/(2n)}\,\Delta(n)\}^2\{\|\tilde{\phi}_n\|_{\alpha,2}^{1/n}/\Delta(n)\}^2$$

by (2.8). The result follows from (2.19) and (2.23).

(b) Let $P_{n-1}\in\mathscr{P}_{n-1}$ satisfy $\|f-P_{n-1}\|_{\Delta(n),\infty}=E_{n-1}[f]$. As is well known, P_{n-1} exists and is unique. Then using $L_n[P_{n-1}]=P_{n-1}$, we have

$$E_{n-1}[f] \leqslant \|f-L_n[f]\|_{\Delta(n),\infty}$$
$$\leqslant \|f-P_{n-1}\|_{\Delta(n),\infty} + \|L_n[f-P_{n-1}]\|_{\Delta(n),\infty}$$
$$\leqslant E_{n-1}[f]\{1+\gamma_{n-1}(2\Delta(n))^{n-1}\, m_0^{1/2}\},$$

by Lemma 3.8(c) applied to $g=f-P_{n-1}$. Now it follows from a result of Erdös and Freud [5, Lemma 2.1, p. 523] that

$$2 \leqslant \limsup_{n} \{\gamma_{n-1}\,\Delta^{n-1}(n)\}^{1/n}$$
$$\leqslant \limsup_{n} \{[\Delta(n)/\Delta(n-1)]/[\|\tilde{\phi}_{n-1}\|_{\alpha,2}^{1/(n-1)}/\Delta(n-1)]\}^{(n-1)/n}$$
$$\leqslant \chi(1)/\mu_-(2)$$

by (2.8), (2.13) and (2.19). The result follows. Q.E.D.

Remarks. (a) The counterexample in Theorem 5.1(a) is more general than that in Theorem 4.1(a′) in that it applies to a larger (and more elegant) class of functions, but the coefficient of $\sigma(f; 1/2)$ is smaller—$\mu_-(2)$ instead of $\nu = \mu_+(2)$.

(b) We next show $\chi(s) = 1$ all $s \geqslant 1$ and $\mu_-(p) > 0$ all $p > 0$ for a class of weights studied by Freud [7]. Freud's weights include the weights $\alpha'(x) = \exp(-|x|^s)$, $x \in (-\infty, \infty)$, $s > 0$.

LEMMA 5.2. *Let $\alpha'(x) = \exp(-2Q(|x|))$ all $x \in \mathbb{R}$, where*

(i) *$Q(x)$ is positive, monotone increasing and continuously differentiable in $(0, \infty)$, and*

(ii) *for some $0 < \eta < 1$, $x^\eta Q'(x)$ is strictly increasing in $(0, \infty)$.*

Let q_x be the root of the equation

$$q_x Q'(q_x) = x \qquad all \quad x \geqslant 0. \tag{5.1}$$

Then

(a) *We may take*

$$\Delta(x) = c q_x \tag{5.2}$$

as our "bounding function" in (2.10) *and* (2.11), *where c is a positive constant.*

(b) $\chi(s) = 1$ *and* $\psi(s) \leqslant s^{1/(1-\eta)}$ *all* $s \geqslant 1$.

(c) $\mu_+(p) \geqslant \mu_-(p) > 0$ *all* $p > 0$ *and* $\nu > 0$.

(d) *Let $f(z)$ be entire and*

$$A = \limsup_{R \to \infty} \log \|f\|_{R,\infty} / Q(R) < \infty. \tag{5.3}$$

Then with c as in (5.2), *and if* $A > 0$,

$$\sigma(f; s) \leqslant c \exp((1-\eta)^{-1})\, \psi(As) \qquad all \quad s > 0. \tag{5.4}$$

When $A = 0$, we may replace $\psi(As)$ by $\lim_{r \to 0+} \psi(r)$ in (5.4). *In particular if*

$$\psi(1/2) = \limsup_{x \to \infty} q_x / q_{2x} < 1 \tag{5.5}$$

then $\sigma(f; 1/2) \to 0$ as $A \to 0$.

Proof. (a) Theorem 1 in Freud [7, p. 49] shows

$$d q_n \leqslant \max\{|x_{nk}|: 1 \leqslant k \leqslant n\} \leqslant c q_n, \qquad n = 1, 2, \ldots$$

where d, c are constants. Further, q_x is strictly increasing in x and $\lim_{x\to\infty} q_x = \infty$, se we may take $\Delta(x) = cq_x$.

(b) Inequality (29) in Freud [7, p. 54] shows

$$q_y/q_x \leqslant (y/x)^{1/(1-\eta)} \qquad \text{all} \quad y \geqslant x > 0$$

(our η is his ρ). Hence it follows from (2.12) and (2.13), respectively, that $\psi(s) \leqslant s^{1/(1-\eta)}$ and $\chi(s) = 1$ all $s \geqslant 1$.

(c) Using inequality (30) in Freud [7, p. 54] and the first line of the proof of Theorem 1 in [7, p. 54], we see that for large n,

$$|x| \leqslant q_{n-1} \leqslant q_{n-1/2} \Rightarrow \alpha'(x) \geqslant \alpha'(1)\exp(-2(1-\eta)^{-1}(2n-1)) \geqslant \delta^{n-1}$$

where $\delta > 0$ is independent of n. Thus for all large enough n,

$$\begin{aligned} \text{meas}\{x\colon \alpha'(x) \geqslant \delta^n\}/\Delta(n) &\geqslant \text{meas}\{x\colon |x| \leqslant q_n\}/\Delta(n) \\ &\geqslant 2/c \end{aligned}$$

by (5.2). Thus (3.7) in Lemma 3.7 holds and so $\mu_+(p) \geqslant \mu_-(p) > 0$ for all $p > 0$. Further, by Lemma 3.8(b), $\nu = \mu_+(2) > 0$.

(d) Erdös and Freud [5, inequality (4.1), p. 530] show that

$$Q(x) \leqslant Q(0) + xQ'(x)/(1-\eta) \qquad \text{all} \quad x > 0. \tag{5.6}$$

Let $f(z) = \sum_{j=0}^{\infty} b_j z^j$ satisfy (5.3). Then for $R \geqslant 1$,

$$\|f\|_{R,\infty} \leqslant \exp[(A + \varepsilon(R))\,Q(R)]\colon \lim_{R\to\infty} \varepsilon(R) = 0.$$

Then by (5.2), (5.6) and Cauchy's estimates for the $\{b_n\}$,

$$\begin{aligned} |b_n|^{1/n}\,\Delta(sn) &\leqslant c\exp[(A + \varepsilon(R))\,Q(R)/n]\,q_{sn}/R \\ &\leqslant c\exp[(A + \varepsilon(R))\{Q(0) + RQ'(R)/(1-\eta)\}/n]\,q_{sn}/R. \end{aligned} \tag{5.7}$$

Assume $A > 0$. If we choose $R = q_{n/A}$ (which minimizes the right member of (5.7) if $\varepsilon(R) = 0$) and if we use (5.1), then we obtain

$$\begin{aligned} \sigma(f; s) &\leqslant c \limsup_{n\to\infty} [(A + \varepsilon(q_{n/A}))\{0 + A^{-1}/(1-\eta)\}]\,q_{sn}/q_{n/A} \\ &\leqslant c\exp((1-\eta)^{-1})\,\psi(As). \end{aligned}$$

When $A = 0$, on chooses $R = q_{n/K(n)}$, where $K(n) \to 0$ as $n \to \infty$. Finally, when $\psi(1/2) < 1$, Lemma 3.2(c) shows $\lim_{r\to 0+} \psi(r) = 0$ and so $\sigma(f; 1/2) \to 0$ as $A \to 0$. Q.E.D.

Remarks. (a) In [1], Aljarrah investigated geometric convergence of Gauss–Jacobi quadrature for Freud's weights subject to the additional restriction (5.5) and based on the size of A in (5.3). Lemma 5.3(d) shows that Aljarrah's results are contained in Theorem 5.1(a). Further, by using $\sigma(f; 1/2)$ rather than A given by (5.3), one does not need to impose (5.5) in studying geometric convergence.

(b) The above Lemma remains valid when for some positive constants K_1, K_2 and some $Q(x)$ satisfying the conditions of Lemma 5.2, we have

$$K_1 \leqslant \alpha'(x)/\exp(-2Q(|x|)) \leqslant K_2 \qquad \text{for all } x \in \mathbb{R}$$

—see Lemma 7 in Freud [6, p. 101].

LEMMA 5.3. *Let $\eta > -1$, $\varepsilon > 0$. Define weights $\alpha_j'(x)$, $j = 1, 2$, by*

$$\alpha_1'(x) = |x|^\eta \exp(-|x|^\varepsilon) \qquad \text{all} \quad x \in \mathbb{R} \tag{5.8}$$

and

$$\alpha_2'(x) = \begin{cases} x^\eta \exp(-x^\varepsilon) & \text{all} \quad x \in (0, \infty) \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

Suppose $\alpha'(x)$ exists in $\mathbb{R}$ except possibly at 0. Suppose for some positive constants c_1, c_2 and $j = 1$ or 2, we have

$$c_1 \alpha_j'(x) \leqslant \alpha'(x) \leqslant c_2 \alpha_j'(x) \qquad \text{all} \quad x \in \mathbb{R}. \tag{5.10}$$

Then for the weight $\alpha'(x)$,

(a) We may take

$$\Delta(x) = cx^{1/\varepsilon} \tag{5.11}$$

as our "bounding function" in (2.10) and (2.11), where c is a positive constant.

(b) $\chi(s) = 1$ and $\psi(s) = s^{1/\varepsilon}$ all $s > 0$.

(c) $\mu_+(p) \geqslant \mu_-(p) > 0$ all $p > 0$ and $\nu > 0$.

Proof. (a) First assume $\alpha'(x) = \alpha_1'(x)$ all $x \in \mathbb{R}$. Freud [6, Theorem B, p. 103] notes that

$$c_{23} n^{1/\varepsilon} \leqslant \max\{|x_{nk}|: 1 \leqslant k \leqslant n\} \leqslant c_{24} n^{1/\varepsilon}, \qquad n = 1, 2, \ldots$$

where c_{23}, c_{24} are independent of n. Thus in this case we may take $\Delta(x) = c_{24} x^{1/\varepsilon}$.

Next suppose $\alpha'(x)=\alpha_2'(x)$ all $x \in \mathbb{R}$. This case is reduced to that for $\alpha_1'(x)$ using a standard trick. The orthonormal polynomials $\{\phi_n\}$ for $\alpha'(x)$ satisfy

$$\int_0^\infty \phi_n(x)\,\phi_m(x)\,x^\eta \exp(-x^\varepsilon)\,dx=\delta_{mn}$$

where δ_{mn} is the Kronecker delta. The substitution $x=y^2$ and evenness of the following integrand yields

$$\int_{-\infty}^\infty \phi_n(y^2)\,\phi_m(y^2)\,|\,y|^{2\eta+1} \exp(-|\,y|^{2\varepsilon})\,dy=\delta_{mn}.$$

Thus if $\{P_n\}$ is the sequence of orthonormal polynomials for the weight $\alpha_3'(y)=|\,y|^{2\eta+1}\exp(-|y|^{2\varepsilon})$, then $P_{2n}(x)=\phi_n(x^2)$, $n=0, 1, 2,...$. Since α_3' is of the form (5.8), we have as before

$$\begin{aligned} c_{23}'(2n)^{1/(2\varepsilon)} &\leqslant \max\{|x|: P_{2n}(x)=0\} \leqslant c_{24}'(2n)^{1/(2\varepsilon)}\\ &\Rightarrow c_{23}'' n^{1/\varepsilon} \leqslant \max\{|x_{nk}|: 1 \leqslant k \leqslant n\} c_{24}'' n^{1/\varepsilon}\end{aligned}$$

as the zeroes x_{nk} of $\phi_n(x)$ are the squares of zeroes of $P_{2n}(x)$. Thus in this case too, we may choose $\Delta(x)$ as in (5.11).

Finally, when $\alpha'(x)$ satisfies only (5.10), Lemma 7 in Freud [6, p. 101] shows that we may still choose $\Delta(x)$ as in (5.11).

(b) follows from (2.12), (2.13) and (5.11).

(c) We are given (5.10). Then there exists $x_0>0$ such that $\alpha'(x) \geqslant \exp(-2x^\varepsilon)$ all $x>x_0$. If $0<\delta<1$,

$$\begin{aligned}\text{meas}\{x: \alpha'(x)\geqslant \delta^n\}/\Delta(n) &\geqslant \text{meas}\ \{x>x_0: \exp(-2x^\varepsilon)\geqslant \delta^n\}/\Delta(n)\\ &=\{[n\,|\log\delta|/2\,]^{1/\varepsilon}-x_0\}/(cn^{1/\varepsilon})\\ &\to [|\log\delta|/2\,]^{1/\varepsilon}/c \qquad \text{as} \quad n\to\infty.\end{aligned}$$

Thus (3.7) holds and $\mu_-(p)>0$ all $p>0$. By Lemma 3.8(b), $\nu=\mu_-(2)>0$.
Q.E.D.

Remarks. (a) For a large class of weights including the Hermite and Laguerre weights, Lemma 5.3 shows that $\mu_-(p)>0$, $\nu>0$ and so the positive and negative assertions of Theorems 4.1, 4.2, 4.3 and 5.1 are applicable to these weights.

(b) Erdös [4] and Freud [5, Remark, p. 531] considered weights $\alpha'(x)=\exp(-2Q(x))$, where $Q(x)$ grows faster than any finite power of x. From the Remark in [5, p. 531] we see $\mu_-(2)\geqslant 1/4$ for these weights.

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